(1)

$$\boxed{y'' - y = 3 - \cos x.}$$

POSSIBLE

$$\boxed{y_g(x) = c_1 e^{x} + c_2 e^{-x} + \tfrac{1}{2}\cos x - 3}$$

$y_g = y_c + y_p$.

$y_c = e^{\lambda x}$

$\lambda^2 - 1 = 0$

$(\lambda+1)(\lambda-1)$

$\lambda = 1, -1$

$$\boxed{y_c(x) = c_1 e^{x} + c_2 e^{-x}}$$

judicial guessing

$$\boxed{y_p \text{ guess: } A + B\cos x + C\sin x.}$$

LI w/ y_c proceed.

$y_p' = 0 + -B\sin x + C\cos x$.

$$\boxed{y_p'' = -B\cos x + -C\sin x.}$$

Plug in y_p'', y_p', y_p into DE

List
cos
sin
const.

$y_p'' - y_p = -B\cos x - C\sin x$

$\quad - B\cos x - C\sin x - A$

$(-2B)\cos x + (-2C)\sin x - A$

$= \text{RHS} = 3 - \cos x.$

Match coefficients of like terms.

$\cos x$ | $-2B = -1$ $\qquad$ $\sin x$ | $-2C = 0$

$\boxed{B = \tfrac{1}{2}}$ $\qquad$ $\boxed{C = 0}$

$A = 3$

$\hookrightarrow \boxed{A = -3}$

(2)] $\boxed{y'' + y' - 6y = 4e^{-3x} + 2e^{2x} + e^{x}}$

TOO LONG
MAYBE, JUST
A sum of
2 exponential.

$y_c(x) = e^{\lambda x}$

$\lambda^2 + \lambda - 6 = 0$

$(\lambda - 2)(\lambda + 3) = 0$

$\lambda = 2, -3$

$y_c(x) = c_1 e^{2x} + c_2 e^{-3x}$

y_p guess: $A e^{-3x} + B e^{2x} + C e^{x}$ Not ready!

Linearly dependent w/ $y_c(x)$.

Box correct y_p just in case

$\boxed{y_p \text{ guess: } A x e^{-3x} + B x e^{2x} + C e^{x}}$ no longer LD. proceed.

$y_p' = A e^{-3x} + Ax(-3e^{-3x}) + B e^{2x} + Bx(2e^{2x}) + C e^{x}$

$y_p'' = -3A e^{-3x} - 3A e^{-3x} - 3Ax(-3e^{-3x}) + 2B e^{2x} + 2B e^{2x} + \cancel{2Bx4e^{2x}}\ 2Bx2e^{2x} + C e^{x}$

plug in y_p'', y_p', y_p into DE.

Before writing, consolidate coefficients of the functions.

List
e^{-3x}
xe^{-3x}
e^{2x}
xe^{2x}
e^{x}

$y_p'' + y_p' - 6y_p = -6A e^{-3x} + 9Axe^{-3x} + 4B e^{2x} + 4Bxe^{2x} + C e^{x}$

$+ A e^{-3x} - 3Axe^{-3x} + B e^{2x} + 2Bxe^{2x} + C e^{x}$

$- 6Axe^{-3x} \qquad - 6Bxe^{2x} - 6C e^{x}$

To prevent errors, organize your functions into columns like this.

$-5A e^{-3x} + 0Axe^{-3x} + 5B e^{2x} + 0Bxe^{2x} - 4C e^{x} = 4e^{-3x} + 2e^{2x} + e^{x}$

Match coefficients

$e^{-3x}\big|$ $-5A = 4$

$A = \frac{4}{-5}$

$xe^{-3x}\big|$ $0A = 0$

A can be $-\frac{4}{5}$.

no contradiction.

$e^{2x}\big|$ $5B = 2$

$B = \frac{2}{5}$

$xe^{2x}\big|$

$4B = 0$.

inconsistent!

check y_p. It is LI w/ y_c.

Find error in derivatives?

Found. $2Bx4e^{2x}$ should be $2Bx2e^{2x}$.

Then $0B = 0$.

B can be $\frac{2}{5}$.

$e^{x}\big|$ $-4C = 1$

$C = -\frac{1}{4}$.

$$\boxed{y_g(x) = c_1 e^{2x} + c_2 e^{-3x} - \frac{4}{5} x e^{-3x} + \frac{2}{5} x e^{2x} - \frac{1}{4} e^{x}}$$

3) $\boxed{y'' + y = \tan x}$ POSSIBLE? Less likely.

↳ need variation of parameters.

$$y_g = y_c + y_p.$$

$$y_c(x) = e^{\lambda x}$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$$\boxed{\begin{array}{l} y_c(x) = c_1 e^{ix} + c_2 e^{-ix} \\ \text{OR} \\ y_c(x) = \tilde{c}_1 \cos x + \tilde{c}_2 \sin x \end{array}}$$

$$f(t) = \frac{\tan(x)}{1}$$

$$\boxed{y_p(x) = u_1 \cos x + u_2 \sin x.}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x) = 1$$

$$u_1'(x) = \frac{-y_2(x) f(x)}{W} = -\frac{\sin x \tan x}{1}$$

$$u_1(x) = -\int \frac{\sin^2 x}{\cos x}\, dx = -\int \frac{1-\cos^2 x}{\cos x}\, dx = -\int \sec x + \int \cos x\, dx.$$

$$u_1(x) = -\int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\, dx + \sin x + \cancel{c}$$

$$u = \sec x + \tan x$$

$$du = (\sec x \tan x + \sec^2 x)\, dx.$$

$$-\int \frac{1}{u}\, du$$

$$\boxed{u_1(x) = -\ln|\sec x + \tan x| + \sin x}$$

$$u_2'(x) = \frac{y_1(x) f(x)}{W} = \frac{\cos x(\tan x)}{1}$$

$$u_2(x) = \int \sin x \, dx = -\cos x + \cancel{C}$$

$$\boxed{u_2(x) = -\cos x}$$

$$y_p(x) = \left(-\ln|\sec x + \tan x| + \sin x\right)\cos x + (-\cos x)\sin x$$

$$\boxed{y_p(x) = -\ln|\sec x + \tan x| \cos x.} \rightarrow \text{sufficient simplification.}$$

or (Trying to get Wolfram alpha form)

$$y_p(x) = \ln\left|(\sec x + \tan(x))^{-1}\right| \cos x$$

$$y_p(x) = \ln\left|\frac{1}{\frac{1}{\cos x} + \frac{\sin}{\cos x}}\right| \cos x$$

$$y_p(x) = \ln\left|\frac{1}{\frac{1+\sin x}{\cos x}}\right| \cos x$$

$$y_p(x) = \ln\left|\frac{\cos x}{1+\sin x}\right| \cos x.$$

$$\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\sin\frac{x}{2} + \cos\frac{x}{2}}$$

$$\frac{1-\cos(x)}{2} = \sin^2\left(\frac{x}{2}\right) = 1 - \cos^2\left(\frac{x}{2}\right)$$

$$\sin x = 2\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)$$

hmm... can't get Wolfram Alpha form.

$$\boxed{y_g = \tilde{c}_1 \cos x + \tilde{c}_2 \sin x + \ln\left|\frac{\cos x}{1+\sin(x)}\right| \cos x.}$$

$$\sin(x) \neq -1$$

$$x \neq -\frac{3\pi}{2} + 2\pi k$$

(4)

15. $y'' + y = 2x\sin x$ → $\boxed{\text{Possible,}}$

$y_g = y_c + y_p$

$y_c(x) = e^{\lambda x}$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

$y_c(x) = C_1 e^{ix} + C_2 e^{-ix}$

OR

$$\boxed{y_c(x) = \tilde{C}_1 \cos x + \tilde{C}_2 \sin x.}$$

y_p guess: $(Ax+B)(C\cos(x) + D\sin(x))$?

$$ACx\cos(x) + ADx\sin(x) + \underbrace{BC\cos(x)} + \underbrace{BD\sin(x)}.$$

Linearly dependent w/ $y_c(x)$.

multiply by x. then LD w/ $x\cos(x)$ and $x\sin(x)$.

multiply by x again.

Relabel product of variables
$AC \to A$, $AD \to B$.

$$\boxed{y_p: Ax\cos(x) + Bx\sin(x) + Cx^2\cos(x) + Dx^2\sin(x).}$$

y_p': $A\cos x + Ax(-\sin x) + B\sin(x) + Bx\cos x + C2x\cos x + Cx^2(-\sin x)$
$+ D2x\sin x + Dx^2(\cos x)$

y_p'': $-A\sin x - Ax\cos x - A\sin x + B\cos x + Bx(-\sin x) + B\cos x + C2\cos x$
$+ C2x(-\sin x) + C2x(-\sin x) - Cx^2\cos x + D2\sin x + D2x\cos x + Dx^2(-\sin x)$
$+ D2x\cos x$

plug in y_p'', y_p', y_p

List of functions

$\cos x$
$\sin x$
$x\cos x$
$x\sin x$
$x^2\cos x$
$x^2\sin x$

$$
\begin{aligned}
y_p'' + y_p &= (2B+2C)\cos x + (-2A+2D)\sin x + (-A+4D)x\cos x + (-B-4C)x\sin x \\
+ \quad & \qquad\qquad\qquad\qquad\qquad\qquad\qquad + A\, x\cos x + Bx\sin x \\
\hline
& (2B+2C)\cos x + (-2A+2D)\sin x + 4D\,x\cos x - 4Cx\sin x
\end{aligned}
$$

$$
\begin{aligned}
& (-C)x^2\cos x + (-D)x^2\sin x \\
+ \quad & +Cx^2\cos x + Dx^2\sin x \\
\hline
& 0 \qquad + 0 \qquad = 2x\sin x
\end{aligned}
$$

Match Terms

$\cos x|$ $2B+2C=0$ $\quad$ $\sin x|$ $-2A+2D=0$

$B=-C$ $\quad$ $A=D$

$x\cos x|$ $4D=0$ $\quad$ $x\sin x|$ $-4C=2$

$\boxed{D=0}$ $\quad$ $\boxed{C=-\frac{1}{2}}$

$\hookrightarrow \boxed{A=0}$ $\quad$ $\hookrightarrow \boxed{B=\frac{1}{2}}$

$$\boxed{y_g(x) = \tilde{c}_1\cos x + \tilde{c}_2\sin x + \frac{1}{2}x\sin x - \frac{1}{2}x^2\cos(x)}$$

15. via Variation of Parameters $(y'' + y = 2x\sin x)$

$$\boxed{\begin{aligned} y_c &= \tilde{c}_1 \cos x + \tilde{c}_2 \sin x \\ y_p &= u_1(x)\cos(x) + u_2(x)\sin(x) \end{aligned}}$$

$$\boxed{f(t) = \frac{2x\sin x}{1}.}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x) = \boxed{1 = W}$$

$$u_1'(x) = -\frac{y_2(x) f(x)}{W} = -\frac{\sin x(2x\sin x)}{1}$$

$$u_1(x) = -2\int x\sin^2 x\,dx = -2\int x\left(\frac{1-\cos(2x)}{2}\right)dx.$$

$$= -2\int\frac{x}{2}\,dx - 2\int -\frac{x\cos 2x}{2}\,dx$$

$$= -\frac{x^2}{2} + \cancel{c}^{\,0} + \int x\cos(2x)\,dx.$$

$u = x \qquad v = \frac{1}{2}\sin(2x)$

$du = dx \qquad dv = \cos(2x)\,dx.$

$$= -\frac{x^2}{2} + \frac{1}{2}x\sin(2x) - \int\frac{1}{2}\sin(2x)\,dx.$$

$$= -\frac{x^2}{2} + \frac{1}{2}x\sin(2x) - \frac{1}{2}\left(\frac{1}{2}\right)(-\cos(2x)) + \cancel{c}^{\,0}$$

$$\boxed{u_1(x) = -\frac{x^2}{2} + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x)}$$

$$u_2'(x) = \frac{y_1(x) f(x)}{W} = \frac{\cos(x)(2x\sin x)}{1}$$

$$u_2(x) = \int x\,2\cos x\sin x\,dx = \int x(\sin 2x)\,dx.$$

$$u_2(x) = \int x(\sin 2x)\,dx.$$

$$u = x \qquad v = -\frac{1}{2}\cos(2x)$$

$$du = dx \qquad dv = \sin(2x)\,dx$$

$$u_2(x) = -\frac{1}{2}x\cos(2x) - \int -\frac{1}{2}\cos(2x)\,dx.$$

$$= -\frac{1}{2}x\cos(2x) + \frac{1}{2}\left(\frac{1}{2}\right)(\sin(2x)) + \cancel{C}$$

$$\boxed{u_2(x) = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x)}$$

$$y_p(x) = \left(-\frac{x^2}{2} + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x)\right)\cos x + \left(-\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x)\right)(\sin x)$$

$$-\frac{x^2}{2}\cos x + \left(\frac{1}{2}x\sin(2x)\cos x - \frac{1}{2}x\cos(2x)\sin x\right) + \frac{1}{4}\left(\cos 2x\cos x + \sin 2x\sin x\right)$$

$$-\frac{x^2}{2}\cos x + \frac{1}{2}x(\sin(2x - x)) + \frac{1}{4}(\cos(2x - x))$$

$$\boxed{y_p(x) = -\frac{x^2}{2}\cos x + \frac{1}{2}x\sin x + \frac{1}{4}\cos x.}$$

↳ Linearly Dependent w/ y_c.
- gets absorbed into $\cos x$ term.

$$\boxed{y_g(x) = \tilde{c}_1\cos x + \tilde{c}_2\sin x - \frac{x^2}{2}\cos x + \frac{1}{2}x\sin x}$$

5

Judicious Guessing

6. $y'' - 8y' + 20y = 100x^2 - 26xe^x$

POSSIBLE if he says use judicious guessing

Variation of Parameters is TOO LONG!

$y_g = y_c + y_p$

$y_c(x) = e^{\lambda x}$

$\lambda^2 - 8\lambda + 20 = 0$

$(\lambda \quad 2)(\lambda \quad 10)$

$\lambda = \dfrac{8 \pm \sqrt{64 - 4(20)}}{2}$ $\quad \begin{array}{r} \cancel{64} \\ 80 \\ -64 \\ \hline 16 \end{array}$

$\lambda = \dfrac{4}{1} \pm \dfrac{\sqrt{-16}}{2} = 4 \pm \dfrac{\sqrt{-1}\sqrt{16}}{2}$

$\lambda = 4 \pm i\dfrac{4}{2}$

$\boxed{\lambda = 4 \pm i2}$

$$\boxed{\begin{gathered} y_c(x) = c_1 e^{(4+2i)x} + c_2 e^{(4-2i)x} \\ \text{OR} \\ y_c(x) = e^{4x}\left(\tilde{c}_1 \cos(2x) + \tilde{c}_2 \sin(2x)\right) \end{gathered}}$$

Note, I can use other variables here like c_2, D_2, B. there is nothing special about the tilde. ~

y_p guess: $Ax^2 + Bx + C + (Dx + E)e^x$

check for terms that are linearly dependent with y_c terms. None.

$y_p' = 2Ax + B + (Dx + E)e^x + De^x$

$y_p'' = 2A + (Dx + E)e^x + De^x + De^x$

plug in y_p, y_p', y_p'' to Diff. Eqn.

$$\left(2A + (Dx+E)e^x + 2De^x\right) - 8\left(2Ax + B + Dxe^x + (E+D)e^x\right)$$
$$+ 20\left(Ax^2 + Bx + C + Dxe^x + Ee^x\right) = 100x^2 - 26xe^x$$
$$+ 0x + 0 + 0e^x$$

$x^2|$ $\quad 20A = 100 \rightarrow A = 5$

$x|$ $\quad -16A + 20B = 0$

$-16(5) + 20B = 0$

$B = \frac{80}{20} = 4 \rightarrow B = 4$

const| $\quad 2A - 8B + 20C = 0$

$2(5) - 8(4) + 20C = 0$

$C = \frac{22}{20} = \frac{11}{10} \qquad C = \frac{11}{10}$

$xe^x|$ $\quad D - 8D + 20D = -26$

$+13D = -26$

$D = -2$

$e^x|$ $\quad E + 2D - 8(E+D) + 20E = 0$

$E + 2(-2) - 8(E-2) + 20E = 0$

$-4 + 16 + 13E = 0$

$E = -\frac{12}{13}$

$$\boxed{y_g(x) = e^{4x}\left(\tilde{c}_1\cos(2x) + \tilde{c}_2\sin(2x)\right) + 5x^2 + 4x + \tfrac{11}{10} + \left(-2x - \tfrac{12}{13}\right)e^x}$$

Same problem w/ variation of parameters.

6. $y'' - 8y' + 20y = 100x^2 - 26xe^x$

$y_g = y_c + y_p$ $\qquad$ $f(x) = \frac{RHS}{1}$

$$\boxed{\begin{array}{l} y_c(x) = e^{4x}\left(\tilde{c}_1 \cos(2x) + \tilde{c}_2 \sin(2x)\right) \text{ from previous soln} \\ y_p(x) = u_1(x) e^{4x}\cos(2x) + u_2(x) e^{4x} \sin(2x) \end{array}}$$

$$W = \begin{vmatrix} e^{4x}\cos(2x) & e^{4x}\sin(2x) \\ e^{4x}(-2\sin(2x)) + 4e^{4x}\cos(2x) & e^{4x}(2\cos(2x)) + 4e^{4x}\sin(2x) \end{vmatrix}$$

$$W = 2(e^{4x}\cos(2x))^2 + 4(e^{4x})^2 \cos(2x)\sin(2x) - \left(-2(e^{4x}\sin(2x))^2 + 4(e^{4x})^2 \sin(2x)\cos(2x)\right)$$

$$W = 2(e^{4x})^2\left(\cos^2(2x) + \sin^2(2x)\right) = 2e^{8x}(1) = \boxed{2e^{8x} = W}$$

$$u_1'(x) = -\frac{y_2(x) f(x)}{W} = -\frac{(e^{4x}\sin(2x))(100x - 26xe^x)}{2e^{8x}}$$

$$= \frac{-100xe^{4x}\sin(2x) + 26xe^{5x}\sin(2x)}{2e^{8x}}$$

$$u_1'(x) = (-50xe^{-4x} + 13xe^{-3x})\sin(2x)$$

not a fun integral

Choose other method!

$$u_1(x) = -50\int x e^{-4x} \underbrace{\sin(2x)}_{u}\,dx + 13\int x e^{-3x}\sin(2x)\,dx.$$

$u = \sin(2x)$ $\quad v = \frac{1}{4}e^{-4x}(-x+1)$

$du = 2\cos(2x)dx$ $\quad dv = xe^{-4x}dx$

$u_2 = x$ $\quad v_2 = -\frac{1}{4}e^{-4x}$

$du_2 = dx$ $\quad dv_2 = e^{-4x}dx$

$-\frac{1}{4}xe^{-4x} - \int -\frac{1}{4}e^{-4x}dx$

$-\frac{1}{4}xe^{-4x} + \frac{1}{4}e^{-4x}$

$$-50\left(-\frac{1}{4}(1-x)e^{-4x}\sin(2x) - \int \frac{1}{4}e^{-4x}(1-x)(2\cos(2x))dx\right) + 13\int xe^{-3x}\sin(2x)dx$$

$$-\frac{1}{2}\int e^{-4x}(1-x)\underbrace{\cos(2x)}_{u}dx$$

$u = \cos(2x)$ $\quad v = \left(-\frac{1}{4}(1-x) + \frac{1}{16}\right)e^{-4x}$

$du = -2\sin(2x)dx$ $\quad dv = e^{-4x}(1-x)\,dx$

$u_2 = (1-x)$ $\quad v_2 = -\frac{1}{4}e^{-4x}$

$du_2 = -dx$ $\quad dv_2 = e^{-4x}dx$

$-\frac{1}{4}(1-x)e^{-4x} - \int -\frac{1}{4}e^{-4x}(-dx)$

$-\int \frac{1}{4}e^{-4x}dx$

$-\frac{1}{4}\left(-\frac{1}{4}\right)e^{-4x}$

$+\frac{1}{16}e^{-4x}$

$$-\frac{1}{2}\left(\left(-\frac{1}{4}(1-x)+\frac{1}{16}\right)e^{-4x}\cos(2x) - \int\left(-\frac{1}{4}(1-x)+\frac{1}{16}\right)e^{-4x}\left(-2\sin(2x)dx\right)\right)$$

Let $I_1 = \int xe^{-4x}\sin(2x)\,dx$

$u_1(x) = -50I_1 + 13\int xe^{-3x}\sin(2x)\,dx.$

$$I_1 = -\frac{1}{4}(1-x)e^{-4x}\sin(2x) - \frac{1}{2}\left(-\frac{1}{4}(1-x)e^{-4x}\cos(2x)\right) - \frac{1}{2}\left(\frac{1}{16}e^{-4x}\cos(2x)\right)$$

$$-\frac{1}{2}\left(-\int\left(-\frac{3}{16}e^{-4x}\right)(-2\sin(2x))dx - \frac{1}{4}(-2)\underbrace{\int xe^{-4x}\sin(2x)dx}_{I_1}\right)$$

$$\left(\frac{1}{2}\right)I_1 = -\frac{1}{4}(1-x)e^{-4x}\sin(2x) + \frac{1}{8}e^{-4x}\cos(2x) + \underbrace{\left(-\frac{1}{8}-\frac{1}{32}\right)}_{\frac{-5}{32}}e^{-4x}\cos(2x)$$

$$+ \frac{3}{16}\int e^{-4x}\sin(2x)\,dx.$$

OK..... I've had enough fun here.